The Green Rings of Pointed, Coserial Hopf Algebras

Abstract

In mathematics, we seek out patterns in the world around us and express these patterns using concrete, well-defined terms. In algebra, we study the internal structures of these objects by looking at their internal symmetries. By doing so, we are able to codify truths in nature into formal language. In fact, such abstract notions as groups and modules can be used to discover results from fields ranging from physics to music.

One important concept in the study of algebra is that of a ring. A *ring* is a collection of objects on which one can add, subtract, and multiply objects. For instance, the set of all numbers on the number line forms a ring, as does the collection of numbers which can be written as fractions, the so-called rational numbers.

One common practice in math is to define mathematical operations such as addition and multiplication in new ways. For example, there is still the usual notion of adding and multiplying numbers. We can even define addition and multiplication between entire collections of numbers, called sets. We can "add" the set of all real numbers to this same set and produce a new set, the set of all pairs (x, y) of real numbers, the coordinate plane. This mathematical definition of addition is not the usual one, but like the usual definition it gives us a way to take two mathematical objects, in this case sets, and produce a new one, here again a set.

A *representation* gives us a way to describe rings using arrays of numbers called matrices. We can actually form a ring whose elements themselves are representations and on which addition and multiplication can be carried out through mathematical operations called direct sums and tensor products. The ring thus formed by taking sums and products of representations is called a *Green ring* or *representation ring*.

In this work, we look at Green rings corresponding to mathematical objects called Hopf algebras. *Hopf algebras* are mathematical sets of interest because as with rings, one can add, subtract, and multiply elements of a Hopf algebra. However, Hopf algebras also have a mathematical operation called comultiplication that gives them very interesting properties, allowing them to be used to model phenomena in areas such as quantum field theory and the Standard Model of particle physics.

The Green rings of many Hopf algebras have been studied in the past. For instance, the Green rings of the group algebras and Taft algebras, two famous examples of Hopf algebras, have been fully classified. However, in most of these previous examples, the Hopf algebras studied have been finite-dimensional, meaning they have a finite basis and have more normal behavior as mathematical objects.

We classify some Green rings of Hopf algebras that are pointed and coserial. Many of these Hopf algebras are in fact infinite-dimensional and thus have more interesting properties. We classify these Green rings using comodules, a type of representation that gives more information about the underlying ring. Hopf algebras satisfying the properties of being pointed and coserial have been previously classified in the literature. However, their Green rings have not been classified and remain an open problem with applications to linear algebra, representation theory, and ring theory.