RESEARCH STATEMENT

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I. Background

My current research focus is in the study of harmonic measure distribution functions. Given a two-dimensional domain Ω, one can place a base point z_0 inside this domain, then allow a particle starting at this point to undergo Brownian motion, a type of random, continuous motion used to model the behavior of particles suspended in a fluid. While moving inside Ω, the particle at some point must cross the boundary of the domain $\partial\Omega$. The **harmonic measure distribution function** (h-function) for this base point z_0 in this domain measures the probability that a particle will first hit the boundary $\partial\Omega$ within a given distance r from the base point. These HMD-functions give information about both the geometric and analytical properties of the domain such as connectedness and curvature of the domain's boundary.

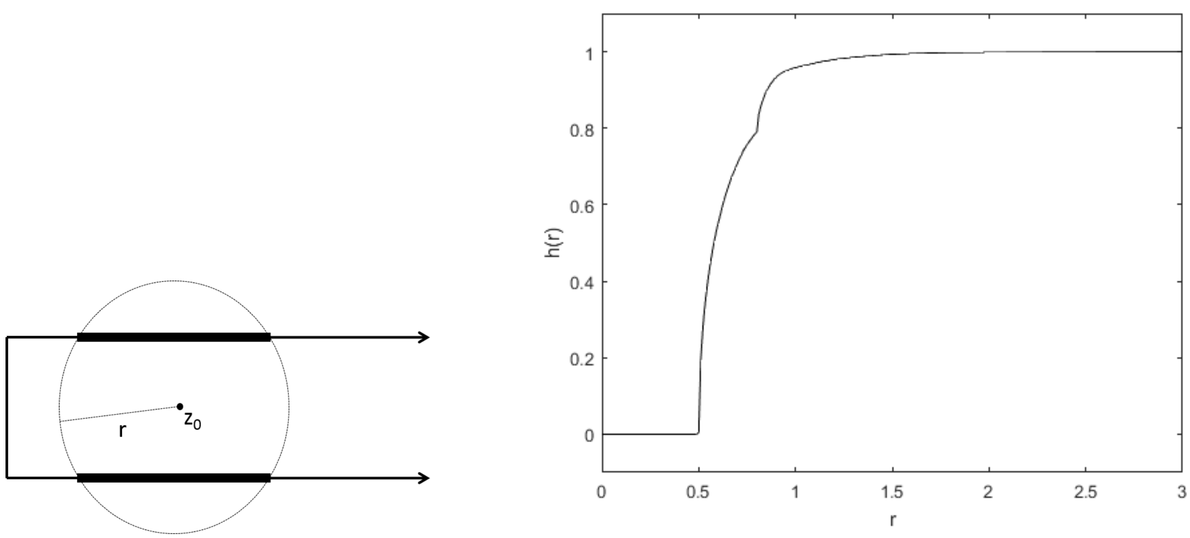

Figure 1: In this half-infinite strip domain Ω to the left, $h(r)$ measures the probability a Brownian particle starting at z_0 first hits the boundary $\partial\Omega$ within distance r from z_0, ie: the bold section of boundary. This function $h(r)$ is plotted to the right.

By a result of Kakutani in [8], this probability function is equivalent to the function representing the harmonic measure of the portion of the boundary $\partial\Omega$ that lies within distance r of z_0. Thus, tools from analysis may be used in many cases to compute an explicit formula for the h-function of a domain. In particular, for a domain Ω that is simply connected, a conformal (angle-preserving map) may be used to map Ω to the unit disc, where the probabilities $h(r)$ may be more easily calculated.

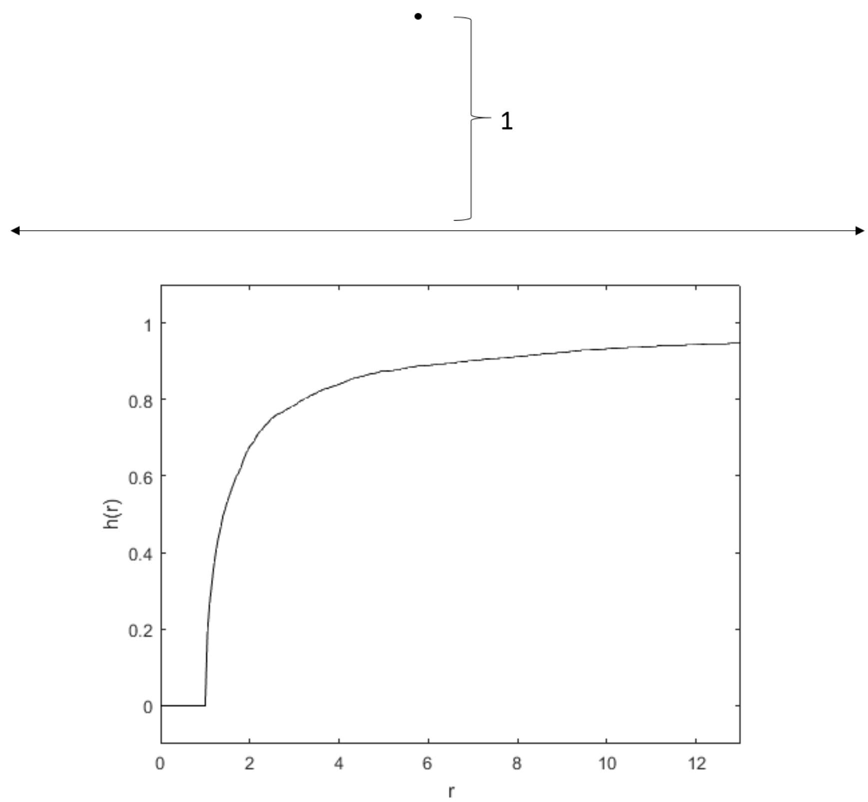

Figure 2: The domain Ω (at the top) consisting of the upper-half plane may be conformally mapped to the inside of a unit disc to compute an explicit formula for its h-function. This h-function is given by

$$h(r) = \begin{cases} 0 & r \leq 1 \\ \frac{2}{\pi} \arccos\left(\frac{1}{r}\right) & r > 1. \end{cases}$$

One class of domains that are particularly important to the study of h-functions is the class of circle domains. A circle domain consists of a disc centered at the base point z_0 with arcs of smaller circles excised from the disc; hence the boundary of such a domain consists of the outer disc as well as these inner arcs of varying radii all centered around z_0.

The reason why circle domains are of particular importance in our study of h-functions is that their h-functions, which are always step functions increasing from 0 to 1, can be used to approximate the h-functions of other domains. In particular, given any function f satisfying necessary properties to be the h-function of some unknown domain, f may be approximated by a sequence of step functions f_i increasing from 0 to 1. By constructing the corresponding

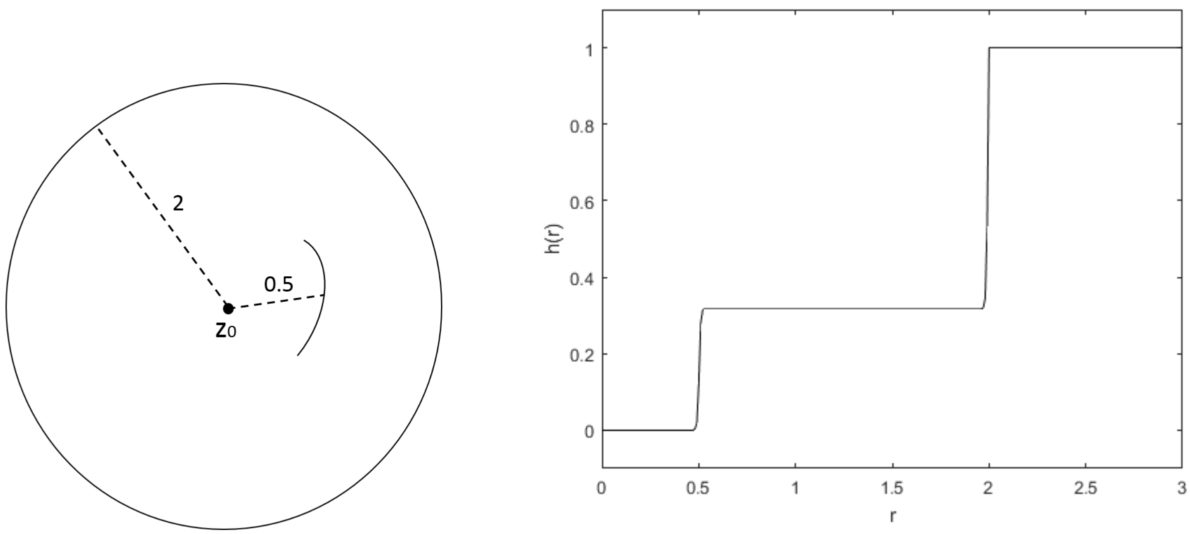

Figure 3: The domain Ω to the left is a circle domain with one inner boundary arc. The corresponding h-function to the right is a step function; while this function may appear to be continuous due to limitations in our program, the actual h-function is discontinuous at $r = 0.5$ and $r = 2$.

circle domains Ω_i for which the step functions f_i are the h-functions, we can ultimately find a domain Ω to which the Ω_i converge and whose h-function will be the function f. This process of constructing domains given an h-function is called the Step Function Program and is outlined in [15].

II. Current Research

I am currently exploring the behavior of h-functions by means of a simulation process called "teleportation." One issue that arises during the simulation of Brownian motion is that there is no upper limit on how long it may take a single Brownian particle to reach the boundary of the domain it starts in. While for most bounded domains it will usually not take long for the particle to cross the boundary in simulation, for unbounded domains such as in Figure 2, the particle can travel very far away from the boundary of the domain before returning to hit the boundary, meaning the simulation of Brownian motion for such domains may be a very long process.

This issue can be resolved by allowing our simulated particles to take *large* steps in a random direction that are of step size equal to the particle's current minimum distance d from the boundary of the domain. We then can calculate the particle's new distance d_1 from the boundary and allow the particle to take a step of size d_1 in a random direction. By repeating this process, the particle will come close to the boundary much faster on average than would a true Brownian particle; however, by repeatedly taking large steps in random directions, we lose none of the randomness inherent in Brownian motion.

Of course, by repeatedly taking steps of size the particle's current distance from the boundary, the particle will typically never reach the boundary in finite time; thus, when the particle comes within some pre-specified fixed distance ϵ of the boundary for some sufficiently small $\epsilon > 0$, the "teleporting" particle is considered to have crossed the boundary for our purposes so that this teleportation process will terminate in finite time.

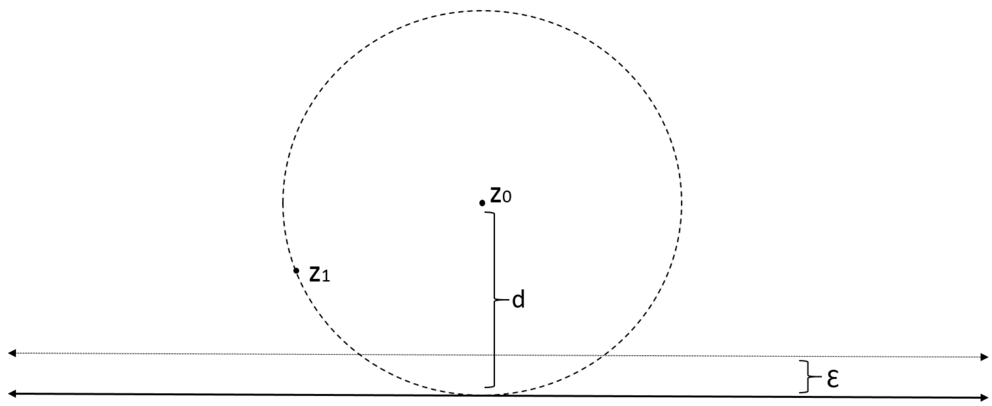

Figure 4: Teleportation is used to simulate random movement in Ω, the upper-half plane. A particle starting at z_0 takes a large random step of size d, the distance from z_0 to $\partial\Omega$, and ends up at z_1. This process is repeated until the particle comes within distance ϵ of the boundary, after which the particle is considered to have crossed $\partial\Omega$.

This past summer, I worked with Oberlin student Miranda Schaum to update older versions of the Brownian motion simulation programs to incorporate teleportation rather than a more accurate representation of Brownian motion. We submitted an article [12] earlier this fall on our findings to the MCURCSM undergraduate research conference. This paper was accepted for publication, and Miranda will present the paper at the conference later this fall. We found that by incorporating teleportation into the programs for simulating h-functions for the upper half-plans and various circle domains, the new programs ran much faster and more efficiently than the old programs, which simulated small random steps. The h-functions produced by both programs were extremely similar (and in the case of the upper-half plane were similar to the actual computed h-function), leading us to conclude that our teleportation simulations produced accurate graphs of the h-functions.

III. Future Work

In the short term, I plan to continue my study of h-functions with Miranda Schaum and another Oberlin student, Yuanzhe Liu, this coming January during Oberlin's winter term. We plan to implement the teleportation method of simulating h-functions in a larger variety of two-dimensional domains, including h-functions for both simply connected domains, whose h-functions can also be calculated directly to compare graphs, as well as for non-simply connected domains, whose h-functions can only be found through simulation. I plan to submit a journal publication this spring based on our results.

Farther in the future, I hope to work with more advanced mathematics students to actually calculate formulas for several h-functions using conformal maps from simply connected domains to the unit disc. These students will then be able to compare h-functions found through analytical means with the h-functions found through simulated data. I also hope to work with students with an interest in statistics and data science to do a more formal analysis of the data produced by the teleportation programs in order to determine the effectiveness of our simulations.

As another long-term goal, I hope to carry out the process described in the Step Function Program: given an appropriate function f satisfying necessary properties to be an h-function, find a two-dimensional domain Ω and base point z_0 in Ω such that f is the appropriate h-function of Ω by approximating f by a sequence of step functions f_i, constructing the circle domains Ω_i such that f_i will be the h-function of Ω_i for each i, then finding the domain Ω to which the circle domains Ω_i converge, which under the appropriate conditions will have f as its h-function. This process will allow us to solve the inverse problem of finding the corresponding domain given an h-function rather than the h-function given the corresponding domain.

IV. Hopf Algebra Research

Finally, I also hope to continue my work done as a graduate student studying the representations of Hopf algebras. While my recent focus in research has been the study of h-functions, I am still very interested in my algebraic research studying ways in which abstract algebraic objects are "represented" using linear transformations of vector spaces. **Hopf algebras** are vector spaces in which several elements may be multiplied to produce one element, but one element can also be *comultiplied* in order to give several elements, creating a more dynamic structure. Hopf algebras are important to many branches of mathematics, having origins in topology and since having spread to geometry, group theory, and representation theory. Of particular note is the role of Hopf algebras in mathematical physics, where they have been used in the Standard Model of Physics to describe how fundamental forces govern interactions between the building blocks of matter.

Hopf algebra representations are of particular interest to mathematicians because while representations of general algebraic structures can be added using direct sums, Hopf algebra representations can also be multiplied via tensor products, creating a ring structure. This structure on finite-dimensional representations of a Hopf algebra is called the **representation ring** or **Green ring** after mathematician J.A. Green. The classification of the Green rings of Hopf algebras is a new, burgeoning area of study in representation theory.

I have studied Hopf algebras of the type

$$H_s = k[x] * k[\mathbb{Z}]/ < xg = \epsilon gx, x^s = 0 >,$$

where g is a generator of $\mathbb{Z}$, k is an algebraically closed field, and ϵ is an s$^{\text{th}}$ root of unity. In my graduate work, I showed that this class of Hopf algebras had Green rings of the form

$$R(H_s) = \mathbb{Z}[T_2, T_3, \cdots, T_s][X, X^{-1}]/I_s$$

where I_s is generated by the following $\frac{s(s-1)}{2}$ relations for $s \geq j \geq i > 1$ where $n = \min\{i, j\}$:

$$T_iT_j = \begin{cases} \displaystyle\sum_{r=0}^{n-1} T_{i+j-(2r+1)}X^r & : i + j \leq s + 1 \\ \displaystyle\sum_{r=0}^{i+j-s-1} T_kX^r + \sum_{r=i+j-s}^{n-1} T_{i+j-(2r+1)}X^r & : i + j > s + 1 \end{cases}$$

In other words, the corresponding Green rings of H_s are quotients of integer polynomial rings, providing us with a way of viewing mathematical interactions between these comodules in a much simpler fashion.

In the future, I plan to continue my study of Hopf algebra representations by extending my results to more general classes of Hopf algebras. In particular, I hope to study the connections between the Green rings of these Hopf algebras in order to determine how different classes of Hopf algebras may interact with each other.

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