Hopf Algebras: Linear Algebra in Action

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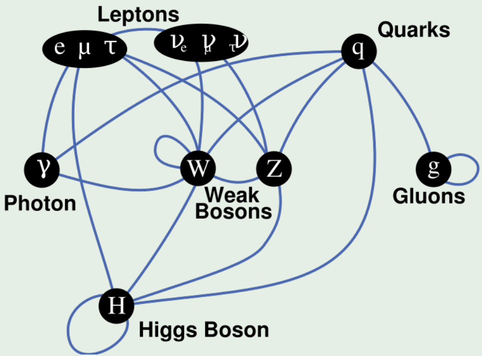
Example

Example 1

The Standard Model of Particle Physics



Fields

Definition 2

A **field** consists of a set k with operations $+, \times$ we call *addition* and *multiplication* satisfying properties in common with the real numbers.

Example 3

$\mathbb{R}$, the set of all real numbers forms a field under the usual addition and multiplication.

Example 4

$\mathbb{C}$, the set of all complex numbers (ie: numbers of the form $a + bi$) forms a field under *complex addition* and *complex multiplication*.

Vector Spaces

Definition 5

Given some field k, a k-**vector space** is a set of objects called **vectors** that may be added to each other and that can be multiplied by elements of k called **scalars** to produce new vectors.

Vector Spaces

Example 6

Given the field $\mathbb{R}$, one $\mathbb{R}$-vector space is the collection of all coordinate pairs (x, y) consisting of ordered pairs $\mathbb{R}^2$ under the operations

- $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- $r \cdot (x, y) = (rx, ry)$

Goal

Given two k-vector spaces V_1 and V_2, we would like to be able to *multiply* these two vector spaces together to form a new vector space.

In doing so, we want to know what happens to individual vectors v_1 and v_2 coming from V_1 and V_2 respectively.

Tensor Products

Suppose that V_1 and V_2 are vector spaces over the same field k.

Definition 7

The **tensor product** of V_1 and V_2 is the k-vector space $V_1 \otimes V_2$ whose elements consist of ordered pairs (v_1, v_2) in $V_1 \times V_2$ subject to the relations:

1. $(v, w_1) + (v, w_2) = (v, w_1 + w_2)$
2. $(t_1, v) + (t_2, v) = (t_1 + t_2, v)$
3. $r \cdot (v_1, v_2) = (r \cdot v_1, v_2) = (v_1, r \cdot v_2)$

Exploring Tensor Products

When using the tensor product structure, we label elements of $V_1 \otimes V_2$ in the form $v_1 \otimes v_2$ rather than (v_1, v_2) to distinguish the tensor product structure.

Example 8

Consider V_1 and V_2 as $\mathbb{R}$-vector spaces. We have the following computations in $V_1 \otimes V_2$:

- $v_1 \otimes w_1 + v_1 \otimes w_2 = v_1 \otimes (w_1 + w_2)$
- $3(v \otimes w_1) - 4(v \otimes w_2) = v \otimes (3w_1) + v \otimes (-4w_2) = v \otimes (3w_1 - 4w_2)$

Exploring Tensor Products

Recall that in $V_1 \otimes V_2$, we have the following relations:

1. $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$
2. $t_1 \otimes v + t_2 \otimes v = (t_1 + t_2) \otimes v$
3. $r \cdot (v_1 \otimes v_2) = (r \cdot v_1) \otimes v_2 = v_1 \otimes (r \cdot v_2)$

Consider arbitrary $\mathbb{R}$-vector spaces V_1 and V_2. Using the above rules, how can we simplify the following expression into a simple tensor?

$$v_1 \otimes v_2 + w_1 \otimes w_2$$

Exploring Tensor Products

1. $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$
2. $t_1 \otimes v + t_2 \otimes v = (t_1 + t_2) \otimes v$
3. $r \cdot (v_1 \otimes v_2) = (r \cdot v_1) \otimes v_2 = v_1 \otimes (r \cdot v_2)$

Example 9

How do we simplify the expression

$$v_1 \otimes w_1 + v_2 \otimes w_2 \ ?$$

We can't!

There are many elements of $V_1 \otimes V_2$ that cannot be written merely as simple tensors. An arbitrary element of $V_1 \otimes V_2$ can be written as a finite sum of such simple tensors.

Goal

Now that we know how to multiply k-vector spaces together through the tensor product, we want to know how to make vector spaces into the enriched structures of *k-algebras* and *k-coalgebras*.

Algebras and coalgebras are the essential tools for understanding the behavior of Hopf algebras in a mathematical setting.

Duality

An important theme in algebra is that of *duality*: mathematical objects have opposites.

For instance, consider the mathematical operation of “add 27”. This mathematical operation has an “opposite” operation that will cancel it out. What is that operation in this case?

Algebras

Let k be a field.

Definition 10

A k-**algebra** (V, M, u) consists of a k-vector space V and the following linear maps:

- $M : V \otimes V \to V$ (called **multiplication**)
- $u : k \to V$ (called the **unit**)

such that M is *associative*:

$$M(M(a \otimes b) \otimes c) = M(a \otimes M(b \otimes c))$$

In this sense, multiplication behaves like multiplication of numbers: one multiplies two elements together to make one element.

Example

Example 11

Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space. By defining the maps

- $M : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R} \qquad M(r \otimes s) = r + s$
- $u : \mathbb{R} \to \mathbb{R} \qquad u(r) = r$

we define an $\mathbb{R}$-algebra structure on $\mathbb{R}$.

Yes, when we create our own mathematical systems, we can *call* an operation "multiplication" that is *actually* addition!

(A different algebra structure can be formed by taking $M(r \otimes s) = rs$.)

Algebra Duality

Consider an algebra. We have the following in an algebra:

- Vector space V
- Multiplication map: $V \otimes V \to V$
- Unit map: $k \to V$
- Associativity condition

How can we create a "dual" structure to an algebra?

Coalgebras

Definition 12

A k-**coalgebra** (V, Δ, ϵ) consists of a k-vector space V and the following linear maps:

- $\Delta : V \to V \otimes V$ (called **comultiplication**)
- $\epsilon : V \to k$ (called the **counit**)

such that Δ is *coassociative*:

$$\text{if } \Delta(v) = a \otimes b, \text{ then } \Delta(a) \otimes b = a \otimes \Delta(b)$$

In this sense, *comultiplication* works like multiplication but backwards: one can *comultiply* one element to make two elements.

Example

Example 13

Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space. By defining the maps

- $\Delta : \mathbb{R} \to \mathbb{R} \otimes \mathbb{R} \qquad \Delta(r) = r \otimes r$
- $\epsilon : \mathbb{R} \to \mathbb{R} \qquad \epsilon(r) = 1$

we define an $\mathbb{R}$-coalgebra structure on $\mathbb{R}$.

Hopf algebras

Definition 14

A **Hopf algebra** V is a k-vector space with the following components:

- A k-algebra structure (V, M, u)
- A k-*co*algebra structure (V, Δ, ϵ)
- An **antipode** map $S : V \to V$, a linear map that is "dual" to the identity map $S : V \to V$

In other words, S cancels out the identity map in some sense.

Example

Example 15

We now consider $\mathbb{R}$, the set of all real numbers as an $\mathbb{R}$-vector space. We have seen that $\mathbb{R}$ forms an $\mathbb{R}$-algebra under the following operations:

- $M : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R} \qquad M(r_1 \otimes r_2) = r_1 + r_2$
- $u : \mathbb{R} \to \mathbb{R} \qquad u(r) = r$

We have also seen that $\mathbb{R}$ forms an $\mathbb{R}$-*co*algebra under the maps

- $\Delta : \mathbb{R} \to \mathbb{R} \otimes \mathbb{R} \qquad \Delta(r) = r \otimes r$
- $\epsilon : \mathbb{R} \to \mathbb{R} \qquad \epsilon(r) = 1$

Now, consider the map $S : \mathbb{R} \to \mathbb{R}$ given by $S(r) = -r$. Then under this map, we find that $\mathbb{R}$ in fact forms an $\mathbb{R}$-Hopf algebra called the **group algebra**.

What do we call the group algebra?

We use two letters to describe the group algebra:

- k – the field we are working over
- G – representing the *group* aspect

So we know that the group algebra is represented by

$$kG$$

What do we call the group algebra?

$$kG = \text{Kevin Gerstle ???}$$

Works Cited

[1] Sorin Dăscălescu, Constantin Năstăsescu, Șerban Raianu. *Hopf Algebras: an Introduction*. Marcel Decker, Inc., 2001.

[2] Christian Kassel. *Quantum Groups*. Springer-Verlag New York, Inc., 1995.

[3] David Radford. *Hopf Algebras*. World Scientific Publishing Co. Pte. Ltd., 2012.