Comprehensive Exam Proposal

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Representation Rings of Hopf Algebras

Abstract

In this talk, we will define the structure of a Hopf algebra, then define the representation ring of a Hopf algebra. We will discuss representation rings of different Hopf algebras, in particular the generalized Taft algebras, then classify the representation ring of a Hopf algebra whose category of comodules is equivalent to the category of k-complexes.

Proposal

Let k be a field. We define a *k-algebra* to be a triple (A, M, u) where A is a k-vector space and $M : A \otimes A \to A$ and $u : k \to A$ are k-vector space morphisms such that

$$M \circ (I \otimes M) = M \circ (M \otimes I)$$

as maps from $A \otimes A \otimes A$ to A and

$$M \circ (u \otimes I) = \sim_1 \qquad , \qquad M \circ (I \otimes u) = \sim_2$$

as maps from where $\sim_1 : k \otimes A \to A$ and $\sim_2 : A \otimes k \to A$ are the canonical isomorphisms. Similarly, we define a *k-coalgebra* to be a triple (C, Δ, ϵ) where C is a k-vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are k-vector space morphisms satisfying

$$(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta$$

as maps from C to $C \otimes C \otimes C$ and

$$(\epsilon \otimes I) \circ \Delta = \sim_3 \qquad , \qquad (I \otimes \epsilon) \circ \Delta = \sim_4$$

as maps from where $\sim_3 : C \to k \otimes C$ and $\sim_4 : C \to C \otimes k$ are the canonical isomorphisms.

A *bialgebra* is a k-vector space H with algebra structure (H, M, u) and coalgebra structure (H, Δ, ϵ) such that M and u are both coalgebra morphisms (or equivalently that

Δ and ϵ are algebra morphisms). If such a bialgebra possesses an *antipode*, or a map $S : H \to H$ satisfying

$$\sum S(h_1)h_2 = \sum h_1 S(h_2) = \epsilon(h)1$$

for $h \in H$, we say H is a *Hopf algebra*. We can then define a *comodule* on H to consist of a k-vector space M and morphism of k-vector spaces $\rho : M \to M \otimes C$ satisfying $(I \otimes \Delta) \circ \rho = (\rho \otimes I) \circ \rho$ and $(I \otimes \epsilon) \circ \rho = \sim_4$. For such a comodule, we write $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes H$.

Given a Hopf algebra H over k, let S be a set of isomorphism classes of finite dimensional indecomposable comodules of H. Let $R(H)$ then be a free abelian group spanned by the set S under direct sums. We note that given H-comodules M and N, $M \otimes N$ forms an H-comodule via the map $\rho : M \otimes N \to (M \otimes N) \otimes H$ given by $\rho(m \otimes n) = \sum (m_0 \otimes n_0) \otimes m_1 n_1$. We can thus introduce multiplication on $R(H)$ by taking

$$[M] \times [N] := [M \otimes N]$$

Under this multiplication, $R(H)$ forms a ring called the *representation ring* or *Green ring* of H. In particular, if we take I to be the set of isomorphism classes of injective indecomposable comodules of H, under the same ring construction we get the *Grothendieck ring* $K(H)$ where $K(H) \leq R(H)$. We note that while this construction can be done with modules of H rather than comodules, comodules are preferable in the infinite dimensional case due to the fact that all objects in H-Comod are locally finite, meaning given any comodule M of H, each element $m \in M$ is contained in some finite dimensional comodule of H. As this is not necessarily true for the modules of H, forming the representation ring using comodules rather than modules is preferable, as we gain more information about H.

These representation rings have been heavily studied in recent years with Hopf algebras[1][3][7]. In particular, mathematicians have classified the representation rings of Taft algebras and generalized Taft algebras, which are somewhat similar in structure to the Hopf algebras we are concerned with except these in particular are finite dimensional [1][7]. In these cases, modules rather than comodules were used to determine the structure of the Green rings.

In this talk, I will go through and explain the above definitions, describe some Hopf algebras for which the representation rings have already been determined, then discuss the structure of the representation ring $R(H_2)$ of the Hopf algebra

$$H_2 = k[x, g, g^{-1}]/\langle xg + gx, x^2 \rangle$$

for a field k satisfying char(k)$\neq 2$. This Hopf algebra is of particular interest due to the fact that as monoidal categories, H_2-Comod$\cong$ k-Comp where H_2-Comod is the category of H_2-comodules and k-Comp is the category of k-complexes [5][8]. It is also of interest that unlike in many previously studied algebras, this Hopf algebra is of infinite dimension, so that in order to study its representation ring, it is more beneficial to look at comodules rather than modules of H_2 [5]. We will discuss the ring structure by showing how tensor products of basis elements can be written in terms of other basis elements, then classify $R(H_2)$ as a quotient of the ring of Laurent polynomials over k adjoined to another variable.

References

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